A class of quasicontractive semigroups acting on Hardy and Dirichlet space

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Abstract

This paper provides a complete characterization of quasicontractive C_0-semigroups on Hardy and Dirichlet space with a prescribed generator of the form $Af = Gf'$. We show that such semigroups are semigroups of composition operators and we give simple sufficient and necessary condition on G. Our techniques are based on ideas from semigroup theory, such as the use of numerical ranges.

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1 Introduction

In this paper, an operator is always assumed to be linear but not necessarily bounded.

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Let X be a Banach space. A one-parameter family $(T(t))_{t\geq 0}$ of bounded linear operators from X to X is a semigroup of bounded linear operators on X if

(i) $T(0) = \mathrm{Id}$, the identity operator on X;

(ii) $T(t+s) = T(t)T(s)$ for every $s, t \geq 0$.

The linear operator A defined by

$$D(A) = \left\{ x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ for } x \in D(A)$$

is the (infinitesimal) generator of the semigroup $T(t)$, $D(A)$ is the domain of A. The semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X is strongly continuous if

$$\lim_{t\downarrow 0} \|T(t)x - x\|_X = 0 \text{ for every } x \in X.$$

Such semigroups are also called C_0-semigroups.

A straightforward consequence of the uniform boundedness theorem is that given a C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X, there exist $w \geq 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{wt} \text{ for all } 0 \leq t < \infty.$$

In the particular case where $M = 1$, the semigroup is said to be quasicontractive. For $M = 1$ and $w = 0$, $(T(t))_{t\geq 0}$ is a semigroup of contractions.

In 1978, Berkson and Porta [5] gave a complete description of the generator A of semigroups of composition operators on the Hardy space $H^2(\mathbb{D})$ (see Section 2.3), induced by a semigroup of analytic self-maps of $\mathbb{D}$ (see Section 2.2 for the definition of such semigroups). Abate [1] rediscovered

the main results of [5], using a different approach, and considering higher dimensions of the scalar space.

Berkson and Porta [5] noticed that such semigroups are strongly continuous on $H^2(\mathbb{D})$ and Siskakis [18, 20] noticed that they are strongly continuous on the Dirichlet space $\mathcal{D}$. Moreover, it is not difficult to see that the generator A of a semigroup of composition operators is of the form $Af = Gf'$. The aim of this paper is to give a complete description of quasicontractive C_0-semigroups of bounded operators on $H^2(\mathbb{D})$ and $\mathcal{D}$ whose generator A is of the form $Af = Gf'$; that is, unlike previous authors, we do not assume *a priori* that we are working with semigroups of composition operators..

The paper is organized as follows. First, in Section 2, we recall the Lumer–Phillips theorem in order to obtain a contractive or quasicontractive C_0-semigroup by means of the numerical range of its generator. We also present the main result of [5] concerning the semigroups of holomorphic functions on $\mathbb{D}$. Then, the weighted Hardy spaces are defined, their main properties are recalled (in particular the fact that some of them are reproducing kernel Hilbert spaces is emphasized) and finally we also study the norm of composition operators induced by a univalent symbol φ on $H^2(\mathbb{D})$ and $\mathcal{D}$. This study is essential to check that the semigroups of composition operators are indeed strongly continous and quasicontractive. To that aim, on the Dirichlet space, the optimal estimates proved in [15] are crucial.

In Section 3, we present our main result on $H^2(\mathbb{D})$, Theorem 3.1, which asserts that the only quasicontractive C_0-semigroups whose generator is of the form $Af = Gf'$ are the semigroups of composition operators. We also prove necessary and sufficient conditions on G, different from the one of Berkson and Porta.

In Section 4, we prove in Theorem 4.3 that the assertions of Theorem 3.1 are equivalent to the fact that A generates a quasicontrative C_0-semigroup on the Dirichlet space, which is itself equivalent to the fact that A generates

a semigroup of composition operators on $\mathcal{D}$.

The last section contains comments on well-known algorithms to test the conditions on G, and explicit examples of constructions of the semigroup of composition operators for a class of analytic polynomial G.

2 General background

2.1 Characterization of contractive C_0-semigroups of bounded operators on a Hilbert space

Besides the well-known Hille–Yosida theorem (see for example Thm. 3.1 in [17]) which characterizes C_0-semigroups in terms of the growth of the resolvent of their generators A, another useful theorem is the Lumer–Phillips theorem (see for example Thm. 4.3 in [17]) which is well-adapted for the characterization of quasicontractive C_0-semigroups in terms of the numerical range of A.

From now on, we assume that the Banach space X on which $T(t)$ is defined is a complex Hilbert space, and we denote it by H. This hypothesis will simplify the definition of dissipative operators involved in the Lumer–Phillips theorem.

Let $A : D(A) \to H$ be a linear operator. Then A is dissipative if

$$\mathrm{Re}\langle Ax, x\rangle \leq 0 \text{ for all } x \in D(A), \|x\| = 1.$$

In other words, $A : D(A) \to H$ is dissipative if the numerical range of A is in the left half-plane.

Theorem 2.1 (Lumer–Phillips) *Let A be a linear operator with dense domain $D(A)$ in X.*

(i) If A is dissipative and there exists $\lambda_0 > 0$ such that $(\lambda_0 \operatorname{Id} - A)D(A) = H$, then A is the generator of contractive C_0-semigroup on H.

(ii) If A is the generator of a contractive C_0-semigroup on H, then A is dissipative and for all $\lambda > 0$, $(\lambda\,\mathrm{Id} - A)D(A) = H$.

This theorem is also of great interest for the characterization of quasicontractive C_0- semigroups observing that $\|T(t)\| \leq e^{wt}$ if and only if $\|\widetilde{T}(t)\| \leq 1$, where $\widetilde{T}(t) := T(t)e^{-wt}$ is the semigroup whose generator is $A - w\,\mathrm{Id}$, if A is the generator of $(T(t))_{t\geq 0}$. In particular we have then the following result.

Corollary 2.2 *Let $A : D(A) \to H$ be a linear operator with a dense domain. Then A generates a quasicontractive C_0-semigroup if and only if $\sup\{\mathrm{Re}\,(\langle Ax, x\rangle) : x \in D(A), \|x\| = 1\} < \infty$ and there exists $\lambda > 0$ such that $(A - \lambda\,\mathrm{Id})D(A) = H$.*

2.2 Semigroups of analytic functions

Theorem 2.3 (Denjoy-Wolff) *Let $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic such that φ is not an elliptic automorphism. Then there is a point $b \in \overline{\mathbb{D}}$ such that φ_n(:=$\varphi \circ \varphi \circ \cdots \varphi$, n times) converges to b uniformly on compact subsets of $\mathbb{D}$.*
If $|b| < 1$, then $\varphi(b) = b$, while if $|b| = 1$, then b behaves as a fixed point in the sense that $\lim_{r\to 1^-} \varphi(rb) = b$. This distinguished point is called the Denjoy–Wolff point of φ.

In the exceptional case of elliptic automorphisms different from the identity map, the sequence of iterates move around an interior fixed point without converging to it.

Definition 2.4 *A one-parameter semigroup of analytic functions of $\mathbb{D}$ into itself is a family $\Phi = \{\varphi_t : t \geq 0\}$ of analytic self-maps of $\mathbb{D}$ such that*

1. *$\varphi_0(z) = z$ for all $z \in \mathbb{D}$;*

2. *$\varphi_{t+s}(z) = \varphi_t \circ \varphi_s(z)$ for all $t, s \geq 0$ and $z \in \mathbb{D}$;*

3. $(t, z) \mapsto \varphi_t(z)$ *is continuous on* $[0, \infty) \times \mathbb{D}$.

Using Vitali's Theorem on convergence of holomorphic functions, it follows that the continuity of $(t, z) \mapsto \varphi_t(z)$ on $[0, \infty) \times \mathbb{D}$ is equivalent to the continuity of $t \mapsto \varphi_t(z)$ for each $z \in \mathbb{D}$. Such semigroups have extensively been studied by Berkson and Porta [5] (see also [21]), who proved the following useful result.

Proposition 2.5 *Let* $\Phi = (\varphi_t)_{t \in \mathbb{R}_+}$ *be a semigroup of analytic functions on* $\mathbb{D}$*, then:*

(i) *For every* $t \in \mathbb{R}_+$*, the function* φ_t *is univalent.*

(ii) *There is a holomorphic mapping* $G : \mathbb{D} \to \mathbb{C}$ *called the generator of* Φ *such that*

$$\frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)) \tag{1}$$

for all $t \geq 0$ *and all* $z \in \mathbb{D}$*. The convergence*

$$G(z) = \lim_{t \to 0^+} \frac{\partial \varphi_t(z)}{\partial t}$$

is uniform on every compact subsets of $\mathbb{D}$.

(iii) *Moreover, the infinitesimal generator* G *of* Φ *has the unique representation*

$$G(z) = F(z)(\overline{\alpha} z - 1)(z - \alpha), \ \forall z \in \mathbb{D},$$

where $F : \mathbb{D} \to \mathbb{C}$ *is analytic and satisfies* $\mathrm{Re}(F) \geqslant 0$*, and* α *is the Denjoy–Wolff point of one (and thus any)* φ_t*,* $t > 0$.

(iv) *Conversely, let* $F : \mathbb{D} \to \mathbb{C}$ *be analytic with* $\mathrm{Re}(F) \geqslant 0$ *and* $\alpha \in \overline{\mathbb{D}}$*, the function* $z \mapsto F(z)(\overline{\alpha} z - 1)(z - \alpha)$*, generates a semigroup of analytic function on* $\mathbb{D}$.

2.3 Operators on weighted Hardy spaces

Our aim is to study semigroups of bounded operators on classical spaces of analytic functions such as the Hardy space $H^2(\mathbb{D})$ and the Dirichlet space $\mathcal{D}$, which are particular cases of the so-called "weighted Hardy spaces".

Definition 2.6 *Take $(\beta_n)_{n\geq 0}$ a sequence of positive real numbers. Then $H^2(\beta)$ is the space of analytic functions*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in the unit disc $\mathbb{D}$ that have finite norm

$$\|f\|_\beta = \left(\sum_{n=0}^{\infty} |c_n|^2 \beta_n^2 \right)^{1/2}.$$

The case $\beta_n = 1$ gives the usual ***Hardy*** *space $H^2(\mathbb{D})$.*
The case $\beta_0 = 1$ and $\beta_n = \sqrt{n}$ for $n \geq 1$ provides the ***Dirichlet*** *space $\mathcal{D}$, which is included in $H^2(\mathbb{D})$.*
The case $\beta_n = 1/\sqrt{n+1}$ produces the ***Bergman*** *space, which contains $H^2(\mathbb{D})$.*

Obviously the polynomials are in $H^2(\beta)$ and with an extra condition on $(\beta_n)_n$, the Hilbert space $H^2(\beta)$ is also a *reproducing kernel space*, i.e. for all $w \in \mathbb{D}$, there exists a function $k_w \in H^2(\beta)$ such that

$$\langle f, k_w \rangle = f(w),$$

for all $f \in H^2(\beta)$ (see p. 19 in [7] and p. 146 in [16]). More precisely, if $(\beta_n)_n$ is such that

$$\sum_{n\geq 0} \frac{|w|^{2n}}{\beta_n^2} < \infty \text{ for all } w \in \mathbb{D}, \tag{2}$$

it follows that $H^2(\beta)$ is a reproducing kernel Hilbert space and

$$k_w(z) = \sum_{n\geq 0} \frac{\overline{w}^n}{\beta_n^2} z^n \qquad \text{with} \qquad \|k_w\|^2_{H^2(\beta)} = \sum_{n\geq 0} \frac{|w|^{2n}}{\beta_n^2}.$$

In fact (2) is also equivalent to the more explicit condition $\liminf(\beta_n)^{1/n} \geq 1$.

Given an operator A (possibly unbounded) defined by $Af(z) = G(z)f'(z)$ on its domain $D(A) = \{f \in H^2(\beta),\ Gf' \in H^2(\beta)\}$ where $G \in H^2(\beta)$, we would like to know if there exists a C_0-semigroup on $H^2(\beta)$ with generator A. The next proposition asserts that two necessary conditions for A to be a C_0-semigroup generator are satisfied.

Proposition 2.7 *Let $(\beta_n)_{n\geq 0}$ a sequence of positive real numbers such that $zH^2(\beta) \subset H^2(\beta)$. Any operator A defined by $Af(z) = G(z)f'(z)$ on $D(A)$ where $G \in H^2(\beta)$ is densely defined on $H^2(\beta)$ and closed.*

Proof: The operator A is defined on polynomials which form a dense family of function in $H^2(\beta)$. We consider a sequence $(f_n : z \mapsto \sum_k a_k^n z^k) \in H^2(\beta)^{\mathbb{N}}$ and two functions $f : z \mapsto \sum_k a_k z^k, g : z \mapsto \sum_k c_k z^k \in H^2(\beta)$ such that $f_n \to f$ and $Gf_n' \to g$ in $H^2(\beta)$. We denote $G(z) = \sum_k b_k z^k$. We now consider the truncated sums, up to the N-th exponent:

$$\|(Gf_n' - Gf')_N\|_2^2 \leqslant \sum_{k=0}^{N} \beta_k \left| \sum_{i+j=k} b_{i-1} j(a_j^n - a_j) \right|^2.$$

As $f_n \to f$ in $H^2(\beta)$, one has

$$\sum_{k=0}^{\infty} \beta_k |a_k^n - a_k|^2 \to 0$$

and thus $\forall k \in \mathbb{N}, |a_k^n - a_k| \to 0$. Hence,

$$\|(g - Gf')_N\|_2 \leqslant \|(g - Gf_n')_N\|_2 + \|(Gf_n' - Gf')_N\|_2 \to 0.$$

We have shown that $\forall k \leqslant N$, $c_k = \sum_{i+j=k} jb_{i-1}a_j$ and since this can be done for each choice of N, we conclude that $g = Gf' \in H^2(\beta)$, $Gf_n' \to Gf'$ in $H^2(\beta)$ and A is closed.

$\square$

Proposition 2.8 *If A is defined by $Af(z) = G(z)f'(z)$ where $G \in H^2(\mathbb{D})$ and $\frac{1}{G} \in H^\infty(\mathbb{D})$, then A cannot be the generator of a one-parameter semigroup.*

Proof: Let λ be a real number; then $\lambda \in \sigma(A)$ if there exists $f \in H^2(\mathbb{D})$ such that

$$G(z)f'(z) = \lambda f(z).$$

Since $\frac{1}{G} \in H^\infty(\mathbb{D})$ the function $u = \int \frac{\lambda}{G} dz$ lies in $H^\infty(\mathbb{D})$ and $f = e^u \in H^\infty \subset H^2(\mathbb{D})$ satisfies $G(z)f'(z) = \lambda f(z)$. Thus $\mathbb{R} \subset \sigma(A)$. This cannot occur for C_0-semigroups, see e.g. [10, Chap.II,1.13].

$\square$

Corollary 2.9 *If A is defined on $H^2(\mathbb{D})$ by $Af(z) = p(z)f'(z)$ where p is polynomial with no roots in the unit closed disc $\overline{\mathbb{D}}$, then A cannot be the generator of a one-parameter semigroup.*

2.4 Bounded composition operators on Hardy and Dirichlet spaces

Composition operators on the Hardy space $H^2(\mathbb{D})$ have a quite surprising property, namely, provided that they are well-defined, they are always continuous. This fact is not true on the Dirichlet space. Moreover, we have the following upper bound for the norm (see Thm. 3.8 in [7]).

Theorem 2.10 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then C_φ maps $H^2(\mathbb{D})$ continuously into $H^2(\mathbb{D})$, and moreover*

$$\|C_\varphi\| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2}.$$

The previous result is very useful to estimate the growth of the norm of semigroups of composition operators on the Hardy space. Indeed, a first consequence of Theorem 2.10 is that each semigroup Φ of analytic functions on $\mathbb{D}$ induces a C_0-semigroup of bounded operators on $H^2(\mathbb{D})$.

Corollary 2.11 *Let $\Phi = (\varphi_t)_{t\geq 0}$ be a semigroup of analytic functions on $\mathbb{D}$. Then $(C_{\varphi_t})_{t\geq 0}$ is a quasicontractive C_0-semigroup on $H^2(\mathbb{D})$.*

Proof: The continuity of $t \mapsto \varphi_t(0)$ implies that $K := \{\varphi_t(0) : 0 \leq t \leq 1\}$ is a compact subset of $\mathbb{D}$. Since G, the generator of Φ, is holomorphic on $\mathbb{D}$, we get

$$\sup_{0\leq t\leq 1} |G(\varphi_t(0))| < \infty.$$

By (1), it follows that there exists $M > 0$ such that $|\varphi_t(0)| \leq Mt$, and then, for $0 \leq t \leq \frac{1}{2M}$, $|\varphi_t(0)| \leq \frac{1}{2}$. Using Theorem 2.10, we also know that

$$\|C_{\varphi_t}\| \leq \left(\frac{1 + |\varphi_t(0)|}{1 - |\varphi_t(0)|}\right)^{1/2},$$

which implies that $\|C_{\varphi_t}\| \leq 1 + O(t)$, and thus there exists $w \geq 0$ such that

$$\|C_{\varphi_t}\| \leq e^{wt},$$

for all $t \geq 0$. Moreover the hypotheses on Φ imply that $C_{\varphi_t} f(z)$ tends to $f(z)$ as t tends to 0, for all $z \in \mathbb{D}$ and all $f \in H^2(\mathbb{D})$. In other words the semigroup $(C_{\varphi_t})_t$ is weakly continuous. It follows that $(C_{\varphi_t})_t$ is strongly continuous (see Thm. I.5.8 in [10]) .

□

On the Dirichlet space, it is not true that C_φ is well defined whenever φ is a self-map of $\mathbb{D}$. For example, for φ an infinite Blaschke product, C_φ is not a bounded composition operator on $\mathcal{D}$. Nevertheless, if φ is univalent, C_φ is bounded on $\mathcal{D}$ (see Section 6.2 of [9]). We have therefore the following preliminary result.

Proposition 2.12 *Let $\Phi = (\varphi_t)_{t\geq 0}$ be a semigroup of analytic functions on $\mathbb{D}$. Then $(C_{\varphi_t})_{t\geq 0}$ is a semigroup of bounded operators on $\mathcal{D}$.*

Using [6], it is possible to prove that Proposition 2.12 is still true for any space $H^2(\beta)$ containing $\mathcal{D}$. But this is beyond the scope of this paper.

3 Quasicontractive semigroups on the Hardy space

From now on, the function G will lie in $H^2(\mathbb{D})$ and the operator A will be defined by $Af = Gf'$ on the domain $D(A) = \{f \in H^2(\mathbb{D}),\ Gf' \in H^2(\mathbb{D})\}$.

Theorem 3.1 *The operator A generates a C_0-semigroup of composition operators on $H^2(\mathbb{D})$ if and only if $\forall z \in \mathbb{D}$,*

$$2\,\mathrm{Re}(\overline{z}G(z)) + \left(1 - |z|^2\right)\mathrm{Re}(G'(z)) \leqslant 0. \tag{3}$$

Proof: Suppose A is such a generator, let (φ_t) denote the corresponding semigroup. From analyticity, one has for small t and fixed z:

$$\varphi_t(z) = z + G(z)t + o(t),$$

$$\varphi_t'(z) = 1 + G'(z)t + o(t).$$

From the Schwarz–Pick lemma (see [3]),

$$|\varphi_t'(z)| \leqslant \frac{1 - |\varphi_t(z)|^2}{1 - |z|^2},$$

and thus,

$$1 + \mathrm{Re}(G'(z))t + o(t) \leqslant \frac{1 - |z|^2}{1 - |z|^2} - \frac{2\,\mathrm{Re}(\overline{z}G(z))}{1 - |z|^2}t + o(t).$$

The condition (3) appears as t tends to 0^+.

We now assume that the condition (3) is valid. For $z_0 \in \mathbb{D}$, consider the initial value problem

$$\frac{dw}{dt} = G(w), \quad w(0) = z_0.$$

Since G is analytic and thus locally Lipschitz, there exist local solutions $w(t) = \varphi_t(z_0)$ by the Cauchy–Peano theorem with values in $\mathbb{D}$. Let

$$\rho(z_1, z_2) = \min_{\gamma(0)=z_1;\gamma(1)=z_2} \int_\gamma \frac{2}{1 - |z|^2}|dz|.$$

So

$$\begin{aligned} \rho(z_0, \varphi_t(z_0)) &\leqslant \int_0^t \frac{2}{1-|\varphi_s(z_0)|^2} \left| \frac{\partial \varphi_s(z_0)}{\partial s} \right| ds \\ &= \int_0^t \frac{2}{1-|\varphi_s(z_0)|^2} \left| G(\varphi_s(z_0)) \right| ds. \end{aligned}$$

Write $f : t \mapsto \dfrac{2}{1-|\varphi_t(z_0)|^2} |G(\varphi_t(z_0))|$, so that

$$\begin{aligned} f'(t) &= \frac{2}{(1-|\varphi_t(z_0)|^2)^2} \left[\frac{\partial |G(\varphi_t(z_0))|}{\partial t} (1-|\varphi_t(z_0)|^2) \right. \\ &\qquad \left. + 2\operatorname{Re}\left(\overline{\varphi_t(z_0)} G(\varphi_t(z_0)) \right) |G(\varphi_t(z_0))| \right] \\ &= \frac{2|G(\varphi_t(z_0))|}{(1-|\varphi_t(z_0)|^2)^2} \left[\operatorname{Re}(G'(\varphi_t(z_0)))(1-|\varphi_t(z_0)|^2) \right. \\ &\qquad \left. + 2\operatorname{Re}\left(\overline{\varphi_t(z_0)} G(\varphi_t(z_0)) \right) \right] \\ &\leqslant 0 \quad \text{by condition (3) at } \varphi_t(z_0). \end{aligned}$$

We conclude that f is a decreasing function, and thus, for $0 \leqslant t_1 < t_2 < \eta$,

$$\rho(\varphi_{t_1}(z_0), \varphi_{t_2}(z_0)) \leqslant (t_2 - t_1) \frac{2|G(\varphi_{t_1}(z_0))|}{1-|\varphi_{t_1}(z_0)|^2}.$$

Therefore, on $[0, \eta)$, φ_t remains in a compact subset of $\mathbb{D}$, so

$$\rho(\varphi_{t_1}(z_0), \varphi_{t_2}(z_0)) \leqslant K|t_2 - t_1|,$$

where K is a constant independent of t_1, t_2 for $0 \leqslant t_1 < t_2 < \eta$. Thus, $\varphi_t(z_0)$ converges as t tends to η. This proves that there exists a solution on $\mathbb{R}_+$ of the initial value problem. Following [5], A generates a C_0-semigroup of composition operators on $H^2(\mathbb{D})$.

$\square$

Remark 3.2 A condition similar to (3) appears in the paper [2], expressed in the language of semi-complete vector fields (semiflows); that is, solutions

to the Cauchy problem

$$\begin{aligned} \frac{du}{dt} + f(u) &= 0, \\ u(0) &= x, \end{aligned}$$

together with the alternative condition

$$\operatorname{Re} f(z)\overline{z} \geq \operatorname{Re} f(0)\overline{z}(1-|z|^2), \qquad z \in \mathbb{D},$$

(see also [19, Prop. 3.5.2]). Thus, as in [5], they start with a semigroup of functions under composition.

Notation 3.3 *For each* $G(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^2(\beta)$*, we write* $\tilde{G}(z) = \alpha_1 + (\alpha_2 + \overline{\alpha_0})z + \sum_{n=3}^{\infty} \alpha_n z^{n-1}$.

An easy test using numerical ranges gives the following necessary condition for the generation of a C_0 semigroup of quasicontractions. A more general result (with a more complicated proof) appears in Proposition 3.7.

Proposition 3.4 *If the operator* A *generates a* C_0*-semigroup of quasicontractions on* $H^2(\beta)$ *with* $\beta = (n^{-\alpha})$ *and* $\alpha \geq 0$*, then*

$$\operatorname*{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re}(\tilde{G}(z)) = \operatorname*{ess\,sup}_{z\in\mathbb{T}} \operatorname{Re}(\overline{z}G(z)) \leqslant 0. \tag{4}$$

Proof: Observing that

$$\sup_{z\in\mathbb{T}} \operatorname{Re}(\overline{z}G(z)) = \sup_{\theta\in\mathbb{R}} \left\{ \operatorname{Re}(\alpha_1) + \operatorname{Re}\left((\overline{\alpha_0} + \alpha_2)e^{i\theta} + \sum_{n=3}^{\infty} \alpha_n e^{i(n-1)\theta} \right) \right\},$$

we can compute the numerical range of A. Let f be an analytic function defined by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $\|f\|_{H^2(\beta)} = 1$ and $f \in D(A)$. Then we

have

$$
\begin{aligned}
\operatorname{Re}\left(\langle G(z)f'(z), f(z)\rangle\right) &= \operatorname{Re}\left(\left\langle \tilde{G}(z)zf'(z), f(z)\right\rangle + \overline{\alpha_0}\sum_{n=0}^{\infty}\beta_n\beta_{n+1}a_n\overline{a_{n+1}}\right) \\
&= \operatorname{Re}(\alpha_1)\sum_{n=0}^{\infty}\beta_n^2 n|a_n|^2 \\
&\quad + \operatorname{Re}\left((\alpha_2+\overline{\alpha_0})\sum_{n=1}^{\infty}\beta_n\beta_{n+1}na_n\overline{a_{n+1}}\right) \\
&\quad + \operatorname{Re}\left(\overline{\alpha_0}\sum_{n=0}^{\infty}\beta_n\beta_{n+1}a_n\overline{a_{n+1}}\right) \\
&\quad + \operatorname{Re}\left(\sum_{k=3}^{\infty}\alpha_k\sum_{n=0}^{\infty}\beta_n\beta_{n+k-1}na_n\overline{a_{n+k-1}}\right).
\end{aligned}
$$

Consider the polynomial functions (obviously in $D(A)$) defined by

$$
f_N(z) = c_N\sum_{n=1}^{N}\frac{\sqrt{6}e^{-in\theta}}{\pi n^{1-\alpha}}z^n,
$$

where c_N is a positive real chosen so that $\|f_N\|_{H^2(\beta)} = 1$. It is clear that c_N tends to 1 as N tends to ∞. Note now that if (4) is not satisfied, then a suitable choice of θ makes $\operatorname{Re}(\langle Af_N, f_N\rangle)$ tend to ∞ as N tends to ∞. It follows that if (4) is not satisfied, then A cannot generate a C_0-semigroup, see e.g. [10, Chap.II, 3.23].

□

Remark 3.5 *It is easy to see that condition (3) implies condition (4).*

Proposition 3.6 *The condition (4) implies the condition (3).*

Proof: Assume G satisfies condition (4) and let $H(z) = z\tilde{G}(z)$. Condition (4) and the maximum principle implies that $\sup_{z\in\mathbb{D}}\tilde{G}(z) \leqslant 0$. Thus by [5, Theorem 3.3] and Corollary 2.11, $f \mapsto Hf'$ generates a C_0-semigroup of composition operators on $H^2(\mathbb{D})$. Hence, by Theorem 3.1, $\forall z\in\mathbb{D}$,

$$
X := 2\operatorname{Re}(\overline{z}H(z)) + (1-|z|^2)\operatorname{Re}H'(z) \leqslant 0.
$$

Now

$$\begin{aligned}
X &= \operatorname{Re}((1+|z|^2)a_1 + 2(\overline{a_0}+a_2)z + \sum_{k=3}^{\infty} a_k z^{k-1}(k-(k-2)|z|^2) \\
&= \operatorname{Re}(2a_0\overline{z} + (1+|z|^2)a_1 + 2a_2 z + \sum_{k=3}^{\infty} a_k z^{k-1}(k-(k-2)|z|^2) \\
&= \operatorname{Re}\left(2\left(a_0\overline{z} + \sum_{k=1}^{\infty} a_k z^{k-1}|z|^2\right) + (1-|z|^2)\sum_{k=1}^{\infty} k a_k z^{k-1}\right) \\
&= 2\operatorname{Re}(\overline{z}G(z)) + (1-|z|^2)\operatorname{Re} G'(z),
\end{aligned}$$

giving condition (3).

$\square$

Proposition 3.7 *Let $(\beta_n)_n$ be a decreasing sequence of positive reals such that $\liminf_{n\to\infty} |\beta_n|^{1/n} \geq 1$ and let $G \in H^2(\beta)$ such that*

$$\operatorname*{ess\,sup}_{w\in\mathbb{T}} \operatorname{Re}(\overline{w}G(w)) > 0.$$

Then

$$\sup \operatorname{Re}\{\langle Af, f\rangle : f \in D(A), \|f\|_{H^2(\beta)} = 1\} = +\infty,$$

where A is defined on $D(A) = \{f \in H^2(\beta) : Gf' \in H^2(\beta)\}$ by $Af = Gf'$.

Before proceeding to the proof, we state the following technical lemma which explains the hypothesis on monotonicity of $(\beta_n)_n$.

Lemma 3.8 *Let $(\beta_n)_n$ be a decreasing sequence of positive reals. Then for all positive integer N, there exists $\eta = \eta(N) > 0$ such that for all $z \in \mathbb{D}$ with $|w| > 1-\delta$, we have*

$$\sum_{n=0}^{N} \frac{|w|^{2n}}{\beta_n^2} < \sum_{n=N+1}^{\infty} \frac{|w|^{2n}}{\beta_n^2}.$$

Proof: Since $(1/\beta_n)_n$ is increasing, we have

$$\sum_{n=0}^{N} \frac{|w|^{2n}}{\beta_n^2} \leq \frac{1}{\beta_N^2}(1+|w|^2+\cdots+|w|^{2N}) = \frac{1}{\beta_N^2}\left(\frac{1-|w|^{2N+2}}{1-|w|^2}\right).$$

On the other hand, we have

$$\sum_{n=N+1}^{\infty} \frac{|w|^{2n}}{\beta_n^2} \geq \frac{1}{\beta_{N+1}^2} \sum_{n=N+1}^{\infty} |w|^{2n} = \frac{|w|^{2N+2}}{\beta_{N+1}^2(1-|w|^2)} \geq \frac{|w|^{2N+2}}{\beta_N^2(1-|w|^2)}.$$

Since $1-|w|^{2N+2} < |w|^{2N+2}$ is equivalent to $|w| > (1/2)^{1/(2N+2)}$, for all $w \in \mathbb{D}$ such that $|w| > \eta(N)$ with $\eta(N) = 1-(1/2)^{1/(2N+2)}$, we have

$$\sum_{n=0}^{N} \frac{|w|^{2n}}{\beta_n^2} < \sum_{n=N+1}^{\infty} \frac{|w|^{2n}}{\beta_n^2}.$$

$\square$

Proof of Proposition 3.7: By hypothesis, there exists $\delta > 0$ and a sequence $(w_k)_k \subset \mathbb{D}$ such that $|w_k| \to 1$ and $\mathrm{Re}(\overline{w_k}G(w_k)) \geq \delta$. Moreover the condition $\liminf_{n\to\infty} |\beta_n|^{1/n} \geq 1$ guarantees that the space $H^2(\beta)$ has reproducing kernels k_w for all $w \in \mathbb{D}$. Now consider the sequence $(\widehat{k_{w_k}})_k$ of normalized reproducing kernels associated with $(w_k)_k$, i.e. $\widehat{k_{w_k}} = \frac{k_{w_k}}{\|k_{w_k}\|_{H^2(\beta)}}$. First assume that $k_{w_k} \in D(A)$. In this case, the remainder of the proof consist in checking that

$$\lim_{k\to\infty} \mathrm{Re}\left(\langle A\widehat{k_{w_k}}, \widehat{k_{w_k}}\rangle_{H^2(\beta)}\right) = +\infty.$$

Note that

$$\langle A\widehat{k_{w_k}}, \widehat{k_{w_k}}\rangle_{H^2(\beta)} = \langle G(\widehat{k_{w_k}})', \widehat{k_{w_k}}\rangle_{H^2(\beta)} = \frac{1}{\|k_{w_k}\|_{H^2(\beta)}^2} G(w_k)k'_{w_k}(w_k),$$

where $k'_{w_k}(z) = \sum_{n\geq 1} \frac{n\overline{w_k}^n}{\beta_n^2} z^{n-1}$. It follows that

$$\langle Ak_{w_k}, k_{w_k}\rangle_{H^2(\beta)} = \sum_{n\geq 1} \frac{nG(w_k)\overline{w_k}|w_k|^{2(n-1)}}{\beta_n^2},$$

and thus

$$\langle A\widehat{k_{w_k}}, \widehat{k_{w_k}}\rangle_{H^2(\beta)} = \frac{\overline{w_k}G(w_k)}{|w_k|^2} \frac{\sum_{n\geq 1} \frac{n|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0} \frac{|w_k|^{2n}}{\beta_n^2}}.$$

Now, for all positive integer N, take $\eta(N)$ as in Lemma 3.8, and k sufficiently large so that $|w_k| > 1 - \eta(N)$. Then we have

$$\begin{aligned} \frac{\sum_{n\geq 1} \frac{n|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0} \frac{|w_k|^{2n}}{\beta_n^2}} &= \frac{\sum_{n=0}^{N} \frac{n|w_k|^{2n}}{\beta_n^2} + \sum_{n=N+1}^{\infty} \frac{n|w_k|^{2n}}{\beta_n^2}}{\sum_{n\geq 0}^{N} \frac{|w_k|^{2n}}{\beta_n^2} + \sum_{n=N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}} \\ &\geq \frac{(N+1)\sum_{n=N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}}{2\sum_{N+1}^{\infty} \frac{|w_k|^{2n}}{\beta_n^2}} = \frac{N+1}{2}. \end{aligned}$$

Therefore, for k sufficiently large (so that $|w_k| > 1 - \eta(N)$), we get

$$\mathrm{Re}\left(\langle A\widehat{k_{w_k}}, \widehat{k_{w_k}}\rangle_{H^2(\beta)}\right) \geq \frac{(N+1)}{2|w_k|^2}\,\mathrm{Re}(\overline{w_k}G(w_k)).$$

Since $\mathrm{Re}(\overline{w_k}G(w_k)) \geq \delta$ and since $|w_k|$ tends to 1, we get the desired conclusion.

If k_{w_k} is not in $D(A)$, the conclusion follows from similar calculation, considering the sequence of polynomials $(k_{w_k}^M)_{M\geq 0}$ defined by

$$k_{w_k}^M = \sum_{n=0}^{M} \frac{\overline{w_k^n}}{\beta_n^2} z^n,$$

which belongs to $D(A)$ and tends to k_{w_k} in $\mathcal{D}$.

□

We are now ready for the main theorem of this section.

Theorem 3.9 *Let $G \in H^2(\mathbb{D})$ and A the operator $f \mapsto Gf'$, defined on the domain $D(A) = \{f \in H^2(\mathbb{D}) : Gf' \in H^2(\mathbb{D})\}$ which is dense in $H^2(\mathbb{D})$. Then the following conditions are equivalent:*
(i) A generates a C_0-semigroup of composition operators on $H^2(\mathbb{D})$;
(ii) $2\,\mathrm{Re}\,\overline{z}G(z) + (1-|z|^2)\,\mathrm{Re}\,G'(z) \leq 0$ *for $z \in \mathbb{D}$;*
(iii) A generates a quasicontractive C_0-semigroup on $H^2(\mathbb{D})$;
(iv) $\mathrm{ess\ sup}_{z\in\mathbb{T}}\,\mathrm{Re}\,\overline{z}G(z) \leq 0$.

Proof: The equivalence between (i) and (ii) is Theorem 3.1. The implication (i)$\Rightarrow$(iii) is Corollary 2.11. The implication (iii)$\Rightarrow$(iv) follows immediately from Proposition 3.7 with $\beta_n = 1$ for all n. Finally, the implication (iv)$\Rightarrow$(ii) is Proposition 3.6.

$\square$

Let A be defined on $D(A) := \{f \in H^2(\mathbb{D}) : Gf' \in H^2(\mathbb{D})\}$ by $Af(z) = G(z)f'(z)$ where $G(z) = \sum_{n=0}^{\infty} \alpha_n z^n$. An easier condition than Condition (3) to test is the following:

$$\operatorname{Re}(\alpha_1) + |\overline{\alpha_0} + \alpha_2| + \sum_{n=3}^{\infty} |\alpha_n| \leqslant 0. \tag{5}$$

In the sequel we present the link between Condition (5) and Condition (3).

Proposition 3.10

(i) Condition (5) implies Condition (3).

(ii) If $G \in \mathbb{C}_2[X]$ (i.e., a polynomial of degree at most 2), then conditions (3) and (5) are equivalent.

(iii) There exists a polynomial function of degree 3 such that condition (3) holds and condition (5) does not.

Proof:

(i) The condition (3) is equivalent to

$$(1+|z|^2)\operatorname{Re}(\alpha_1)+2\operatorname{Re}((\overline{\alpha_0}+\alpha_2)z)+\sum_{n=3}^{\infty}\operatorname{Re}(\alpha_n((2-n)|z|^2+n)z^{n-1}) \leqslant 0.$$

The condition (5) is

$$\operatorname{Re}(\alpha_1) + |\overline{\alpha_0} + \alpha_2| + \sum_{n=3}^{\infty} |\alpha_n| \leqslant 0,$$

that is,

$$(1+|z|^2)\operatorname{Re}(\alpha_1)+(1+|z|^2)|\overline{\alpha_0}+\alpha_2|+\sum_{n=3}^{\infty}(1+|z|^2)|\alpha_n|\leqslant 0.$$

Note that $2\operatorname{Re}((\overline{\alpha_0}+\alpha_2)z)\leqslant 2|\overline{\alpha_0}+\alpha_2||z|\leqslant(1+|z|^2)|\overline{\alpha_0}+\alpha_2|$. On the other hand, the arithmetico-geometric inequality gives, $\forall k\in\mathbb{N}^*,\ \forall x\in\mathbb{R}_+$,

$$\frac{1+x^2+(k-1)x^{k+2}}{k+1}\geqslant\sqrt[k+1]{1\times x^2\times x^{(k+2)(k+1)}}=x^k,$$

i.e.

$$1+x^2\geqslant x^k((k+1)-(k-1)x^2).$$

We now observe that

$$\operatorname{Re}(\alpha_n((2-n)|z|^2+n)z^{n-1})\leqslant|\alpha_n||z^{n-1}|((2-n)|z|^2+n)\leqslant(1+|z|^2)|\alpha_n|.$$

(ii) Let $G(z)=\alpha_0+\alpha_1 z+\alpha_2 z^2$.
If condition (3) is true, we have in particular

$$\forall\theta\in\mathbb{R},\ 2\operatorname{Re}(e^{-i\theta}G(e^{i\theta}))\leqslant 0\quad\text{ i.e. }\quad\operatorname{Re}(\alpha_1+(\overline{\alpha_0}+\alpha_2)e^{i\theta})\leqslant 0.$$

For $\theta=-\arg(\overline{\alpha_0}+\alpha_2)$, we get

$$\operatorname{Re}(\alpha_1)+|\overline{\alpha_0}+\alpha_2|\leqslant 0.$$

(iii) Take $G(z)=-z+\frac{z^2}{\sqrt{3}}-\frac{z^3}{\sqrt{3}}$. Note that G does not satisfy condition (5). On the other hand, note that $G(z)=-z(1-\frac{z}{\sqrt{3}}+\frac{z^2}{\sqrt{3}})=-zF(z)$. In [5], it is shown that, if $\operatorname{Re}(F)\geqslant 0$, then G generates a C_0-semigroup of composition operators on $H^2(\beta)$, which is equivalent to condition (3). Since $\operatorname{Re}(F)$ satisfies the maximum principle, for $h(\theta):=\operatorname{Re}(F(e^{i\theta})=1-\frac{1}{\sqrt{3}}\cos(\theta)+\frac{1}{\sqrt{3}}\cos(2\theta)$, F maps the disc into the right-half plane if h is nonnegative. For that purpose, note that

$$h'(\theta)=\frac{1}{\sqrt{3}}\sin(\theta)(1-4\cos(\theta)).$$

It follows that $h'(\theta) = 0 \Leftrightarrow \theta = 0$ or $\theta = \pi$ or $\cos(\theta) = \frac{1}{4}$. A direct computation gives: $h(0) = 1$, $h(1) = 1 + \frac{2}{\sqrt{3}}$ and if $\cos(\theta) = \frac{1}{4}$, $h(\theta) = 1 - \frac{9}{8\sqrt{3}} > 0$. Therefore G satisfies condition (3).

□

4 Quasicontractive semigroups on the Dirichlet space

Recall that the Dirichlet space norm is defined by

$$\|f\|_{\mathcal{D}}^2 = |a_0|^2 + \sum_{k=1}^{\infty} k|a_k|^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z), \tag{6}$$

for $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and it is induced by an inner product that may be written, at least formally, as

$$\langle f, g \rangle_{\mathcal{D}} = \langle f, zg' \rangle_{H^2(\mathbb{D})} + f(0)\overline{g(0)}.$$

Proposition 4.1 *For $G \in \mathcal{D}$ and A the operator $f \mapsto Gf'$, defined on the domain $D(A) = \{f \in \mathcal{D} : Gf' \in \mathcal{D}\}$, which in dense in $\mathcal{D}$, the following two conditions are equivalent:*
(i) $\operatorname{ess\,sup}_{z \in \mathbb{T}} \operatorname{Re} \overline{z} G(z) \leq 0$;
(ii) $\sup\{\operatorname{Re}\langle Af, f \rangle_{\mathcal{D}} : f \in D(A), \|f\|_{\mathcal{D}} = 1\} < \infty$.

Proof: Suppose that $\operatorname{ess\,sup}_{z \in \mathbb{T}} \operatorname{Re} \overline{z} G(z) \leq 0$. Then

$$\begin{aligned} \operatorname{Re}\langle Af, f \rangle_{\mathcal{D}} &= \operatorname{Re}\langle Gf', zf' \rangle_{H^2(\mathbb{D})} + \operatorname{Re}\left(G(0)f'(0)\overline{f(0)} \right) \\ &= \operatorname{Re}\left(\frac{1}{2\pi} \int_0^{2\pi} G(z)\overline{z}|f'(z)|^2 \, d\theta \right) + \operatorname{Re}\left(G(0)f'(0)\overline{f(0)} \right), \end{aligned}$$

with $z = e^{i\theta}$, and the supremum of this quantity over $\|f\|_{\mathcal{D}} = 1$, $f \in D(A)$ is clearly finite.

Conversely, suppose that ess sup $\operatorname{Re} \overline{z}G(z) > 0$. By considering an (outer) function u with $|u| = 1$ on a set of positive measure where $\operatorname{Re} \overline{z}G(z) > \delta > 0$ and $|u| = 1/2$ on its complement (see Thm. 4.6 in [12]) we see that $\liminf_{n\to\infty} \operatorname{Re}\langle Gu^n, zu^n\rangle_{H^2(\mathbb{D})} > 0$. It now follows that there is a function $f \in D(A)$ with $\langle f, f\rangle_{\mathcal{D}} = 1$ and $\operatorname{Re}\langle Gf', zf'\rangle_{H^2(\mathbb{D})} > 0$.

Now define a sequence $(f_k)_k$ in $\mathcal{D}$ by setting $f_k' = z^k f'$ and $f_k(0) = 0$.

So if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$f_k(z) = \sum_{n=1}^{\infty} a_n \frac{n}{n+k} z^{n+k},$$

and hence

$$\begin{aligned} \langle f_k, f_k\rangle_{\mathcal{D}} &= \sum_{n=1}^{\infty} |a_n|^2 \left(\frac{n}{n+k}\right)^2 (n+k) \\ &= \sum_{n=1}^{\infty} |a_n|^2 \frac{n^2}{n+k} \le \|f\|_{\mathcal{D}}^2, \end{aligned}$$

and thus this tends to zero by the dominated convergence theorem.

Now

$$\operatorname{Re}\langle Af_k, f_k\rangle_{\mathcal{D}} = \operatorname{Re}\langle \overline{z}Gf_k', f_k'\rangle_{H^2(\mathbb{D})} = \operatorname{Re}\langle Gf', f'\rangle_{H^2(\mathbb{D})}$$

On normalizing the functions f_k we see that

$$\sup\{\operatorname{Re}\langle Af, f\rangle_{\mathcal{D}} : f \in \mathcal{D}, \|f\|_{\mathcal{D}} = 1\} = \infty.$$

□

Proposition 4.2 *Let $G \in \mathcal{D}$ and A the operator $f \mapsto Gf'$, defined on the domain $D(A) = \{f \in \mathcal{D} : Gf' \in \mathcal{D}\}$, which is dense in $\mathcal{D}$. If A generates a C_0-semigroup of composition operators on $\mathcal{D}$, then this semigroup is quasicontractive.*

Proof: Given a semigroup $(C_{\varphi_t})_{t\geq 0}$ acting on $\mathcal{D}$, we must show that $\|C_{\varphi_t} f\|_{\mathcal{D}} = \|f\|_{\mathcal{D}}(1+O(t))$ for small $t>0$. First, since φ_t is injective, we have the well-known inequality

$$\begin{aligned}\int_{\mathbb{D}} |(f\circ\varphi_t)'(z)|^2\,dA(z) &= \int_{\mathbb{D}} |(f'\circ\varphi_t(z))|^2|\varphi_t'(z)|^2\,dA(z)\\ &= \int_{\varphi(\mathbb{D})} |f'(w)|^2\,dA(w) \leq \int_{\mathbb{D}} |f'(w)|^2\,dA(w),\end{aligned}$$

taking $w=\varphi(z)$. Therefore the composition operator C_{φ_t} is bounded on $\mathcal{D}$. Moreover, by [15, Thm. 2],

$$\|C_{\varphi_t}\| \leq \sqrt{\frac{L+2+\sqrt{L(4+L)}}{2}},$$

where $L=\log\left(\frac{1}{1-|\varphi_t(0)|^2}\right)$. This upper bound is sharp since it is an equality whenever $\mathbb{D}\setminus\varphi_t(\mathbb{D})$ is of Lebesgue area measure equal to 0.

Siskakis [20] proved that, as in the case of the Hardy space, A is of the form $A(f)=G(z)f'(z)$, where G is an holomorphic function on $\mathbb{D}$ and $\varphi_t(z)=z+G(z)t+o(t)$. It follows that, for $t\to 0$,

$$\|C_{\varphi_t}\| \leq 1+O(t),$$

since $L=O(t^2)$. Therefore, there exists $w>0$ such that $\|C_{\varphi_t}\|\leq e^{wt}$ for all $t\geq 0$, and thus $(C_{\varphi_t})_{t\geq 0}$ is then a quasicontractive C_0-semigroup.

$\square$

Theorem 4.3 *Let $G\in\mathcal{D}$ and A the operator $f\mapsto Gf'$, defined on the domain $D(A)=\{f\in\mathcal{D} : Gf'\in\mathcal{D}\}$, which is dense in $\mathcal{D}$. Then the following conditions are equivalent:*

(i) A (extended to its natural domain in $H^2(\mathbb{D})$) generates a C_0-semigroup of composition operators on $H^2(\mathbb{D})$;

(ii) $2\operatorname{Re}\overline{z}G(z)+(1-|z|^2)\operatorname{Re}G'(z)\leq 0$ for $z\in\mathbb{D}$;

(iii) A (extended to its natural domain in $H^2(\mathbb{D})$) generates a quasicontractive

C_0-semigroup on $H^2(\mathbb{D})$;
(iv) ess $\sup_{z\in\mathbb{T}} \operatorname{Re} \overline{z}G(z) \leq 0$*;*
(v) $\sup\{\operatorname{Re}\langle Af, f\rangle_{\mathcal{D}} : f \in D(A), \|f\|_{\mathcal{D}} = 1\} < \infty$*;*
(vi) A generates a quasicontractive C_0-semigroup on $\mathcal{D}$;
(vii) A generates a C_0-semigroup of composition operators on $\mathcal{D}$.

Proof: Conditions (i)–(iv) have already been shown to be equivalent in Theorem 3.9. The equivalence of conditions (iv) and (v) is shown in Proposition 4.1. For (i)⇒(vii) is detailed in [20]. The fact that (vii)⇒(vi) is given in Proposition 4.2. Finally, (vi)⇒(v) by Lumer–Phillips result (see Corollary 2.2).

□

5 Comments

In [5], as well as in Condition 5, the description of the generator of a C_0-semigroup of composition operators relies on analytic functions F or $\tilde{G}$ which map $\mathbb{D}$ into the right or left half-plane. For that purpose, let us recall the Carathéodory–Toeplitz theorem [4, 18].

Theorem 5.1 (Carathéodory–Toeplitz) *Let $f(z) = \sum_{n=0}^{\infty} \mu_n z^n$ and consider for $k \geqslant 1$ the matrices $M_k = (m_{i,j})_{1\leqslant i,j\leqslant k}$ where $m_{i,j} = \mu_{j-i}$ if $i \leqslant j$ and $m_{i,j} = 0$ otherwise. Then f maps the disc to the right half plane if and only if the Hermitian matrix $N_k = M_k + \overline{M_k'}$ is nonnegative definite for all $k \geqslant 1$.*

This theorem has to be considered with the Sylvester Criterion.

Theorem 5.2 (Sylvester Criterion) *Let $A = \left(a_{ij}\right)_{1\leqslant i,j\leqslant n}$ be Hermitian. Then A is positive definite if and only if the n matrices $A_p = \left(a_{ij}\right)_{1\leqslant i,j\leqslant p}$ with $1 \leqslant p \leqslant n$ have positive determinant.*

Here is an example where we can use those tools.

Example 5.3 *Let $G(z) = a_0 + a_1 z + a_2 z^2 \in \mathbb{C}_2[X]$, thanks to condition (4), we have that G generates a C_0-semigroup of analytic functions on $\mathbb{D}$ if and only if* $\sup(\tilde{G}(\mathbb{T})) \leqslant 0$. *Besides,* $\sup(\tilde{G}(\mathbb{T})) < 0$ *if and only if*

$$\det\big(-\operatorname{Re}(a_1)\big) > 0, \quad \det\begin{pmatrix} -\operatorname{Re}(a_1) & -(\overline{a_0} + a_2) \\ -(\overline{a_2} + a_0) & -\operatorname{Re}(a_1) \end{pmatrix} > 0$$

i.e.

$$\begin{cases} \operatorname{Re}(a_1) < 0 \\ \operatorname{Re}(a_1)^2 - |\overline{a_0} + a_2|^2 = (\operatorname{Re}(a_1) - |\overline{a_0} + a_2|)(\operatorname{Re}(a_1) + |\overline{a_0} + a_2|) > 0 \end{cases}$$

i.e.

$$\operatorname{Re}(a_1) + |\overline{a_0} + a_2| < 0.$$

We have recovered condition (5).

One may wonder if the quasicontractive C_0-semigroup whose generator is given by A can be determined on $H^2(\mathbb{D})$ or $\mathcal{D}$. We know that it is a semigroup of composition operators C_{φ_t}, with

$$\frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)).$$

This is an important and not so easy issue, which can be answered in some particular cases, as follows. Those examples are slight generalizations of the ones presented in [21].

(i) If $G(z) = az + b$ with $a \neq 0$, we have

$$\varphi_t(z) = e^{at} z + \frac{b}{a}(e^{at} - 1).$$

Furthermore, the Denjoy–Wolff point α of this holomorphic semigroup is $\alpha = -\frac{b}{a} \in \mathbb{D}$.

(ii) When G *is a polynomial of degree* 2, defined by $G(z) = c(z-a)(z-b)$:

- If $a \neq b$, we get

$$\varphi_t(z) = \frac{z(ae^{bct} - be^{act}) + ab(e^{act} - e^{bct})}{z(e^{bct} - e^{act}) + (ae^{act} - be^{bct})},$$

whose Denjoy–Wolff point is $\alpha = a \in \mathbb{D}$ if $\operatorname{Re} a < \operatorname{Re} b$ and $\alpha = b \in \mathbb{D}$ if $\operatorname{Re} a > \operatorname{Re} b$. In the case where $\operatorname{Re} a = \operatorname{Re} b$ it happens that $\varphi_{t_n} = Id$ for $t_n = \frac{2\pi n}{\operatorname{Im} a - \operatorname{Im} b}$, so φ_t is an automorphism.

- If $a = b$, we find another expression for φ_t:

$$\varphi_t(z) = \frac{z(1 - act) + a^2 ct}{-zct + (1 + act)},$$

whose Denjoy–Wolff point is $\alpha = a$.

(iii) As G *is polynomial of higher degree*, we usually do not have explicit expression of the semigroup (φ_t). Yet, some cases can be found:

- If $G(z) = c(z - a)^n$ then $\forall t \in \mathbb{R}_+, \forall z \in \mathbb{D}$,

$$\varphi_t(z) = a + \frac{z - a}{(1 - nct(z - a)^{n-1})^{\frac{1}{n-1}}}.$$

Note that, if $c = 1$, the only possible case is when $a = 1$.

- If $G(z) = cz(z^n - a)$ then $\forall t \in \mathbb{R}_+, \forall z \in \mathbb{D}$,

$$\varphi_t(z) = \frac{ze^{-ct}}{\left(1 - z^n\left(\frac{1 - e^{-nct}}{a}\right)\right)^{\frac{1}{n}}}.$$

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References

[1] M. Abate, *The infinitesimal generators of semigroups of holomorphic maps.* Ann. Mat. Pura Appl. 161 (1992), no. 4, 167–180.

[2] D. Aharonov, S. Reich, and D. Shoikhet, Flow invariance conditions for holomorphic mappings in Banach spaces. *Math. Proc. R. Ir. Acad.* 99A (1999), no. 1, 93–104.

[3] L.V. Ahlfors, *Conformal invariants: topics in geometric function theory.* McGraw-Hill Series in Higher Mathematics. McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[4] M. Bakonyi and T. Constantinescu, *Schur's algorithm and several applications*. Pitman Research Notes in Mathematics Series, 261. Longman Scientific & Technical, Harlow, 1992.

[5] E. Berkson and H. Porta, *Semigroups of analytic functions and composition operators*. Michigan Math. J. 25 (1978), no. 1, 101–115.

[6] I. Chalendar and J.R. Partington, Norm estimates for weighted composition operators on spaces of holomorphic functions. *Complex Anal. Oper. Theory* 8 (2014), no. 5, 1087–1095.

[7] C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[8] E.B. Davies, *Linear operators and their spectra*. Cambridge Studies in Advanced Mathematics, 106. Cambridge University Press, Cambridge, 2007.

[9] O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A primer on the Dirichlet space*. Cambridge Tracts in Mathematics, 203. Cambridge University Press, Cambridge, 2014.

[10] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*. Springer-Verlag, New York, 2000.

[11] E.A. Gallardo-Gutiérrez and J.R. Partington, Norms of composition operators on weighted Hardy spaces. *Israel J. Math.* 196 (2013), no. 1, 273–283.

[12] J.B. Garnett, *Bounded analytic functions*. Graduate Texts in Mathematics, 236. Revised first edition. Springer, 2010.

[13] R.A. Horn and C.R. Johnson, *Matrix analysis*. Second edition. Cambridge University Press, Cambridge, 2013.

[14] W. König, Semicocycles and weighted composition semigroups on H^p. *Michigan Math. J.* 37 (1990), no. 3, 469–476.

[15] M.J. Martin and D. Vukotić, Norms and spectral radii of composition operators on the Dirichlet space. *J. Math.Anal. Appl.* 304 (2005), 22–32.

[16] J.R. Partington, *Interpolation, identification, and sampling*. London Mathematical Society Monographs. New Series, 17. The Clarendon Press, Oxford University Press, New York, 1997.

[17] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*. Applied mathematical Sciences, 44. Springer, 1983.

[18] I. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, *J. Reine Angew. Math.* 147 (1917), 205–232. English translation in: *I. Schur Methods in Operator Theory and Signal Processing*, I. Gohberg, ed., Birkhäuser, 1986, 31–89.

[19] D. Shoikhet, *Semigroups in geometrical function theory*. Kluwer Academic Publishers, Dordrecht, 2001.

[20] A.G. Siskakis, Semigroups of composition operators on the Dirichlet space. *Results Math.* 30 (1996), 165–173.

[21] A.G. Siskakis, Semigroups of composition operators on spaces of analytic functions, a review. *Studies on composition operators (Laramie, WY, 1996)*, 229–252, Contemp. Math., 213, Amer. Math. Soc., Providence, RI, 1998.